The Spectral Dimension of Generic Trees

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Abstract We define generic ensembles of infinite trees. These are limits as $N \to \infty$ of ensembles of finite trees of fixed size N, defined in terms of a set of branching weights. Among these ensembles are those supported on trees with vertices of a uniformly bounded order. The associated probability measures are supported on trees with a single spine and Hausdorff dimension $d_h = 2$. Our main result is that the spectral dimension of the ensemble average is $d_s = 4/3$, and that the critical exponent of the mass, defined as the exponential decay rate of the two-point function along the spine, is $1/3$.

1 Introduction

Diffusion on random geometric structures has received considerable attention in recent years. The motivation comes from a wide range of different areas of physics such as: percolation theory where the percolation clusters provide fluctuating geometries; the physics of random media, where the effect of impurities is often modeled by a random geometry, see e.g. [1]; and quantum gravity, where space-time itself is treated as a fluctuating manifold, see e.g. [2]. In particular, the long time characteristics of diffusion have been studied for the purpose of providing quantitative information on the mean large scale behavior of the geometric objects in question. The *spectral dimension* is one of the simplest quantities which provides such information.

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In this article the geometric structures under consideration are tree graphs with a distinguished vertex r, called the root. The spectral dimension d_s is given by

$$p(t) \sim t^{-d_s/2} \quad \text{for } t \to \infty, \tag{1}$$

where $p(t)$ denotes the return probability for a simple random walk starting at r as a function of (discrete) time t, averaged with respect to the given probability distribution of graphs. In Sect. 4 below we calculate d_s in terms of the singularities of the generating function for the sequence $p(t), t \in \mathbb{N}$. In many random geometry applications it is in fact the generating function which is the main interest since it is more closely related to the Laplacian.

There is another natural notion of dimension for random geometries, the *Hausdorff dimension* d_h, defined as

$$V(R) \sim R^{d_h}, \tag{2}$$

where $V(R)$ is the ensemble average of the volume of a ball of radius R. In general it is easier to evaluate the Hausdorff dimension and we will see that it is 2 for all the ensembles studied in this paper. For fixed graphs the spectral and Hausdorff dimensions are related by

$$d_h \geq d_s \geq \frac{2d_h}{1 + d_h}, \tag{3}$$

provided both exist [3]. This relation is also satisfied in some examples of random geometries [4].

The exact value of d_s is only known for a rather limited class of models. For bond percolation on a hypercubic lattice the value of d_s for the incipient infinite cluster at criticality is unknown, but it is conjectured to be 4/3 in sufficiently high dimensions [5].[1] For planar random surfaces related to two-dimensional quantum gravity the spectral dimension is likewise unknown, but conjectured on the basis of numerical simulations and scaling relations to be 2 [7–9]. In [10] it is shown that the spectral dimension of a limit of rooted planar graphs with randomly chosen root cannot exceed 2 if the order of vertices is uniformly bounded. For recent simulations investigating the spectral dimension in higher dimensional gravity, see [11, 12].

In a preceding article [4] we developed techniques for analyzing a particular class of random geometries, called *random combs*, which are special tree graphs composed of an infinite linear chain, called the *spine*, to which a number of linear chains, called *teeth*, are attached according to some probability distribution. In particular, we determined the spectral dimension as well as other critical exponents for various random combs. The techniques of [4], however, are not strong enough to deal with general models of random trees, not to mention other models of random graphs. The main purpose of this article is to reinforce these methods thus enabling us to determine the spectral dimension of a large class of random tree models. This is the class of *generic infinite tree ensembles* which we define in the next section. Among these ensembles are those supported on trees with vertices of a uniformly bounded order. Our main result is that the spectral dimension of generic infinite tree ensembles is

$$d_s = 4/3. \tag{4}$$

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[1]After the completion of this paper a proof appeared of the Alexander–Orbach conjecture for oriented percolation in dimensions $d > 6$ [6].

Included in this class are some models that have been considered previously, see [13–15]. Early results of Kesten [16] suggested the scaling (4). In a recent article [17] it is proven for critical percolation on a Cayley tree that (1) holds almost surely for individual infinite clusters (not only for the average which we consider) with $d_s = 4/3$, up to logarithmic factors. Adapting the techniques of [17], similar results should be obtainable for the models considered here. The results of the present paper, dealing with averaged quantities, provide a complementary perspective. We believe that our method of proof is conceptually very simple, in addition to being applicable to a large class of random tree models.

This article is organized as follows. In Sect. 2 we define the models of trees that will be considered and describe in some detail their probability distribution, which is supported on trees with one infinite spine. In Sect. 3 we explain the connection between our ensembles of trees and trees that are generated by a Galton–Watson process. This connection allows us to use some well known results about such processes to analyze the trees. Section 4 contains a proof of (4). In Sect. 5 we introduce the critical exponent of the mass and prove that it equals 1/3 for generic infinite trees. This means that diffusion along the spine of the generic infinite trees is anomalous as is discussed in detail for random combs in [4]. Finally, Sect. 6 contains a few concluding remarks on possible extensions and open problems. Some technical results are relegated to two appendices.

2 Generic Ensembles of Infinite Trees

An ensemble of random graphs is a set of graphs equipped with a probability measure. In this section we define the ensembles of trees to be investigated in this paper. We start by defining a probability measure on finite trees and show that it yields a limiting measure on infinite trees.

Let Γ be the set of all planar rooted trees, finite or infinite, such that the root, r, is of order (or valency) 1 and all vertices have finite order. If $\tau \in \Gamma$ is finite we let $|\tau|$ denote its size, i.e. the number of links in τ, and the subset of Γ consisting of trees of fixed size N will be called Γ_N. The subset consisting of the infinite trees will be denoted Γ_∞. Given a tree $\tau \in \Gamma$, the ball $B_R(\tau)$ of radius R around the root is the subgraph of τ spanned by the vertices whose graph distance from r is less than or equal to R. Note that $B_R(\tau)$ is again a rooted tree. It is useful to define the distance $d_\Gamma(\tau, \tau')$ between two trees τ, τ' as $(R+1)^{-1}$, where R is the radius of the largest ball around r common to τ and τ'. We shall view Γ as a metric space with metric d_Γ, see [18] for some of its properties. For $\tau \in \Gamma$ we let $\tau \setminus r$ denote the set of all vertices in τ except the root.

Given a set of non-negative *branching weights* w_n, $n \in \mathbb{N}$, we define the *finite volume partition functions*, Z_N, by

$$Z_N = \sum_{\tau \in \Gamma_N} \prod_{i \in \tau \setminus r} w_{\sigma_i}, \tag{5}$$

where N is a positive integer and σ_i denotes the order of vertex i. We assume $w_1 > 0$, since Z_N vanishes otherwise, and we also assume $w_n > 0$ for some $n \geq 3$ since otherwise only the linear chain of length N would contribute to Z_N. Under these assumptions the generating function g for the branching weights,

$$g(z) = \sum_{n=1}^{\infty} w_n z^{n-1}, \tag{6}$$

is strictly increasing and strictly convex on the interval $[0, \rho)$, where ρ is the radius of convergence for the series (6), which we assume is positive.

It is well known, see e.g. [2], that the generating function for the finite volume partition functions,

$$Z(\zeta) = \sum_{N=1}^{\infty} Z_N \zeta^N, \tag{7}$$

satisfies the equation

$$Z(\zeta) = \zeta g(Z(\zeta)). \tag{8}$$

The analytic function $Z(\zeta)$ which vanishes at $\zeta = 0$ is uniquely determined by (8). Letting ζ_0 denote the radius of convergence of the series (7), the limit

$$Z_0 = \lim_{\zeta \uparrow \zeta_0} Z(\zeta) \tag{9}$$

is finite and $\leq \rho$. In the following we consider the case where

$$Z_0 < \rho. \tag{10}$$

This is the condition on the branching weights which singles out the generic ensembles of infinite trees to be defined below. In particular, all sets of branching weights with infinite ρ define a generic ensemble.

Assuming (10) the value of Z_0 is determined as the unique solution to the equation

$$Z_0 g'(Z_0) = g(Z_0) \tag{11}$$

and ζ_0 can then be found from (8). Taylor expanding g around Z_0 in (8) and using (11) yields the well known singular behavior of Z at ζ_0,

$$Z(\zeta) = Z_0 - \sqrt{\frac{2g(Z_0)}{\zeta_0 g''(Z_0)}} \sqrt{\zeta_0 - \zeta} + O(\zeta_0 - \zeta). \tag{12}$$

We shall need the following result on the asymptotic behavior of Z_N, the proof of which can be found in [19], Sects. VI.5 and VII.2.

Lemma 1 *Under the stated assumptions on the branching weights and assuming* (10) *the asymptotic behavior of Z_N is given by*

$$Z_N = \sqrt{\frac{g(Z_0)}{2\pi g''(Z_0)}} N^{-\frac{3}{2}} \zeta_0^{-N} (1 + O(N^{-1})), \tag{13}$$

provided the integers n for which $w_{n+1} \neq 0$ have no common divisor > 1. Otherwise, if $d \geq 2$ denotes their largest common divisor, we have

$$Z_N = d \sqrt{\frac{g(Z_0)}{2\pi g''(Z_0)}} N^{-\frac{3}{2}} \zeta_0^{-N} (1 + O(N^{-1})), \tag{14}$$

if $N \equiv 1 \bmod d$, and $Z_N = 0$ otherwise.

We define the probability distribution ν_N on Γ_N by

$$\nu_N(\tau) = Z_N^{-1} \prod_{i \in \tau \backslash r} w_{\sigma_i}, \quad \tau \in \Gamma_N, \tag{15}$$

provided $Z_N \neq 0$. Using Lemma 1 the existence of a limiting probability measure ν on Γ can be established by a minor modification of the arguments in [18], where the existence was proven for the *uniform trees* corresponding to the weight factors $w_n = 1$ for all n. We state the result in the following theorem, providing an outline of the proof in Appendix 1.

Theorem 1 *Viewing ν_N as a probability measure on Γ we have, under the same assumptions as in Lemma 1, that*

$$\nu_N \to \nu \quad as\ N \to \infty, \tag{16}$$

where ν is a probability measure on Γ concentrated on the subset Γ_∞.

Here the limit should be understood in the weak sense, meaning that

$$\int_\Gamma f d\nu_N \to \int_\Gamma f d\nu \tag{17}$$

for all bounded functions f on Γ which are continuous in the topology defined by the metric d_Γ. Moreover, N is restricted to values such that $Z_N \neq 0$. We call the ensembles (Γ, ν) *generic ensembles of infinite trees*, referring back to the genericity assumption (10). The expectation w.r.t. ν will be denoted $\langle \cdot \rangle_\nu$.

There is a simple description of ν which is analogous to the description provided in [18] for the measure on uniform trees. Given an infinite tree τ a *spine* is an infinite linear chain (non-backtracking path) in τ starting at the root. The result that ν is concentrated on the subset of trees with a single spine is of crucial importance; it enables us to assume that all infinite trees have a unique spine. We denote the vertices on the spine by $s_0 = r, s_1, s_2, \ldots,$ ordered by increasing distance from the root. To each s_n, $n \geq 1$, are attached a finite number of *branches*, i.e. finite trees in Γ, by identifying their roots with s_n. If s_n is of order σ there are $\sigma - 2$ branches attached to the spine at s_n. Let $T_1', \ldots, T_k'$ denote those to the left (relative to the direction from r along the spine) and $T_1'', \ldots, T_\ell''$ those to the right, ordered clockwise around s_n, see Fig. 1. It follows from (122) in Appendix 1 that the probability that s_n has $k \geq 0$ left branches and $\ell \geq 0$ right branches is

$$\varphi(k, \ell) = \zeta_0 w_{k+\ell+2} Z_0^{k+\ell}, \tag{18}$$

for all vertices s_n on the spine. Note that this probability only depends on $k + \ell = \sigma_{s_n} - 2$. Finally, the individual branches at s_n are independently distributed, and for each of them the probability that a particular finite tree T occurs is

$$\mu(T) = Z_0^{-1} \zeta_0^{|T|} \prod_{i \in T \backslash r} w_{\sigma_i}. \tag{19}$$

We note that μ is the grand canonical distribution with fugacity ζ_0.

Fig. 1 The first few vertices on
the spine of a tree and the finite
branches attached
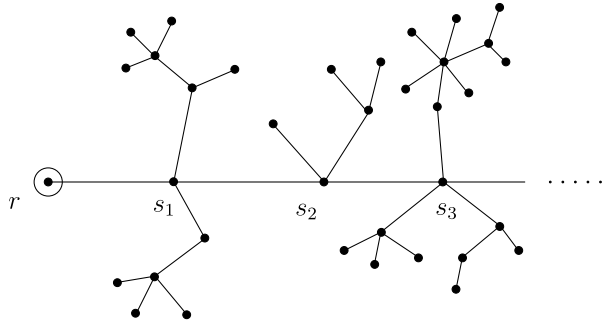
3 Generic Trees and Galton–Watson Processes

The probabilities $\mu(T)$ defined in (19) can be viewed as arising from a critical *Galton–Watson process* as we show in Lemma 2 below. A Galton–Watson (GW) process is specified by a sequence p_n, $n = 0, 1, 2, \ldots$, of non-negative numbers which are called *offspring probabilities* and satisfy

$$\sum_{n=0}^{\infty} p_n = 1. \tag{20}$$

We say that the process is *critical* if the mean number of offspring is 1, i.e.,

$$\sum_{n=1}^{\infty} n p_n = 1. \tag{21}$$

A critical GW process gives rise to a probability distribution π on the subset of finite trees T in Γ, see e.g. [20], by

$$\pi(T) = \prod_{i \in T \setminus r} p_{\sigma_i - 1}. \tag{22}$$

Lemma 2 *Suppose the branching weights w_n correspond to a generic ensemble of infinite trees. Consider the probability distribution μ (19) on the set of finite trees in Γ. Then μ corresponds to a critical Galton–Watson process with offspring probabilities*

$$p_n = \zeta_0 w_{n+1} Z_0^{n-1}. \tag{23}$$

Proof With p_n given by (23) we get

$$\sum_{n=0}^{\infty} p_n = \zeta_0 \sum_{n=0}^{\infty} w_{n+1} Z_0^{n-1} = \zeta_0 Z_0^{-1} g(Z_0) = 1, \tag{24}$$

where the last equality follows from (8). Furthermore, by (22),

$$\pi(T) = \zeta_0^{|T|} \prod_{i \in T \setminus r} w_{\sigma_i} Z_0^{\sigma_i - 2} = Z_0^{-1} \zeta_0^{|T|} \prod_{i \in T \setminus r} w_{\sigma_i}, \tag{25}$$

since

$$\sum_{i \in T \setminus r} (\sigma_i - 2) = -1, \tag{26}$$

for a tree T with a root of order 1. This proves that $\pi(T) = \mu(T)$. Finally,

$$\sum_{n=1}^{\infty} n p_n = \zeta_0 \sum_{n=1}^{\infty} n w_{n+1} Z_0^{n-1} = \zeta_0 g'(Z_0), \tag{27}$$

which proves, in view of (8) and (11), that the process is critical. $\qquad\square$

In the following we let f denote the generating function for the offspring probabilities given by (23),

$$f(z) = \sum_{n=0}^{\infty} p_n z^n = \zeta_0 \sum_{n=1}^{\infty} w_n Z_0^{n-2} z^{n-1} = \zeta_0 Z_0^{-1} g(Z_0 z). \tag{28}$$

Equations (20) and (21), or equivalently (8) for $\zeta = \zeta_0$ and (11), may then be written

$$f(1) = 1 \quad \text{and} \quad f'(1) = 1, \tag{29}$$

respectively. Moreover, the genericity assumption (10) is equivalent to *assuming f to be analytic in a neighborhood of the unit disc.*

If T is a finite tree, let $h(T)$ denote its *height*, i.e. the maximal distance from the root r to a vertex in T. As a consequence of Lemma 2 we note the following result.

Lemma 3 *Let μ be the measure on finite trees given by (19) and let $\langle \cdot \rangle_\mu$ denote the expectation w.r.t. μ. Then*

$$\mu(\{T \in \Gamma \,|\, h(T) > R\}) = \frac{2}{f''(1)R} + O(R^{-2}) \tag{30}$$

for R large. Moreover,

$$\langle |B_R| \rangle_\mu = R. \tag{31}$$

Proof Both properties are standard consequences of the fact that μ is a critical GW process, see e.g. [20], Sects. I.5 and I.10. $\qquad\square$

For $\tau \in \Gamma$ let $D_R(\tau)$ denote the number of vertices in τ at distance R from the root. The relation between ν and μ described above gives rise to the following useful result.

Lemma 4 *Let (Γ, ν) be a generic ensemble of infinite trees with corresponding critical Galton–Watson measure μ. For any bounded function u on Γ such that $u(\tau) = u(B_R(\tau)), \tau \in \Gamma$, i.e. $u(\tau)$ depends only on the part of τ contained in the ball of radius R around the root, we have*

$$\int_\Gamma u(\tau) d\nu(\tau) = \int_\Gamma u(T) D_R(T) d\mu(T). \tag{32}$$

Proof Let $\tau \in \Gamma$ and let T denote the finite subtree of τ consisting of the first $R+1$ vertices $s_0, s_1, s_2, \ldots, s_R$ on the spine together with all the branches attached to these vertices. Then $u(\tau) = u(T)$. Let $\Gamma(R)$ denote the set of all rooted finite planar trees with a marked vertex s, of order one, at a distance R from the root. Then we can write the left hand side of (32) as a sum over $\Gamma(R)$ by performing the integration over the branches attached to the vertex s_R on the spine including the infinite one containing the part of the spine beyond s_R, which is distributed according to ν. The result is

$$\sum_{T \in \Gamma(R)} u(T) \prod_{i \in T \setminus \{r, s_R\}} p_{\sigma_i - 1}. \tag{33}$$

On the other hand, the right hand side of (32) is a sum over finite rooted trees, where only trees of height at least R contribute because of the factor $D_R(T)$. This sum can be replaced by a sum over finite rooted trees with one marked vertex at distance R from the root upon deleting the factor $D_R(T)$. Summing over the branches containing the descendants of the marked vertex again yields the sum (33) thereby establishing the lemma. $\square$

The integration formula (32) has the following application which will be needed in the next section.

Lemma 5 *Let (Γ, ν) be a generic ensemble of infinite trees. Then there exists a constant $c > 0$ such that*

$$\langle |B_R|^{-1} \rangle_\nu \leq c R^{-2} \tag{34}$$

for all $R \geq 1$.

Proof Using the function

$$u(\tau) = \begin{cases} D_R(\tau)^{-1} & \text{if } D_R(\tau) \neq 0, \\ 0 & \text{if } D_R(\tau) = 0, \end{cases} \tag{35}$$

in Lemma 4 we get from (30) that

$$\langle D_R^{-1} \rangle_\nu \leq \frac{c'}{R} \tag{36}$$

for $R \geq 1$, where c' is a positive constant. From this we conclude

$$\langle |B_R(\tau)|^{-1} \rangle_\nu = \left\langle \frac{1}{D_1 + \cdots + D_R} \right\rangle_\nu \leq R^{-1} \langle (D_1 D_2 \ldots D_R)^{-\frac{1}{R}} \rangle_\nu$$

$$\leq R^{-1} \prod_{i=1}^{R} \langle D_i^{-1} \rangle_\nu^{\frac{1}{R}} \leq c' R^{-1} (R!)^{-\frac{1}{R}} \leq c R^{-2}. \tag{37}$$

$\square$

Remark Lemma 5 can easily be strengthened to

$$c_1 R^{-2} \leq \langle |B_R|^{-1} \rangle_\nu \leq c_2 R^{-2}, \tag{38}$$

where c_1 and c_2 are positive constants. Indeed, by Jensen's inequality, we have

$$\langle |B_R|^{-1} \rangle_\nu \geq (\langle |B_R| \rangle_\nu)^{-1}, \tag{39}$$

and from standard arguments using generating functions one has

$$\langle |B_R| \rangle_\nu = \frac{1}{2} f''(1) R(R-1) + R. \tag{40}$$

Equation (40) also shows that the Hausdorff dimension, see (2), of the ensemble (Γ, ν) is 2.

4 The Spectral Dimension

For $\tau \in \Gamma$ let ω be a random walk on τ starting at the root at time 0. Let $\omega(t)$ denote the vertex where ω is located after t steps, $t \leq |\omega|$. The generating function for return probabilities of simple random walk on τ, $Q_\tau(x)$, is given by

$$Q_\tau(x) = \sum_{\omega: r \to r} (1-x)^{\frac{1}{2}|\omega|} \prod_{t=1}^{|\omega|-1} \sigma_{\omega(t)}^{-1}, \quad 0 < x \leq 1, \tag{41}$$

where the sum is over all walks in τ starting and ending at the root r, including the trivial walk consisting of r alone which contributes 1 to $Q_\tau(x)$. The generating function for first return probabilities $P_\tau(x)$ is given by (41) except that the sum excludes the trivial walk and is restricted to walks that do not visit r in between the initial and final position. The functions $Q_\tau(x)$ and $P_\tau(x)$ are related by the identity

$$Q_\tau(x) = \frac{1}{1 - P_\tau(x)}. \tag{42}$$

This equation implies that $P_\tau(x) \leq 1$. There is a more general recurrence relation which we will use repeatedly in the rest of the paper. Let $\tau \in \Gamma$ and v the vertex next to the root. Let $\tau_1, \tau_2, \ldots, \tau_k \in \Gamma$ be the subtrees of τ meeting the link (r, v) at v. By decomposing walks from the root and back into a sequence of excursions into the different branches τ_i of the tree τ we find

$$P_\tau(x) = \frac{1-x}{k+1 - \sum_{i=1}^k P_{\tau_i}(x)}. \tag{43}$$

For the ensemble (Γ, ν) we set

$$Q(x) = \langle Q_\tau(x) \rangle_\nu \tag{44}$$

and define the critical exponent α associated with $Q(x)$ by

$$Q(x) \sim x^{-\alpha} \tag{45}$$

as $x \to 0$. By $a(x) \sim x^\beta$ for $x \to 0$ we mean that for any $\epsilon > 0$ there are constants c_1 and c_2, which may depend on ϵ, such that for x small enough

$$c_1 x^{\beta+\epsilon} \leq a(x) \leq c_2 x^{\beta-\epsilon}. \tag{46}$$

This gives a precise meaning to the definition of the spectral dimension (1) and, assuming it exists, d_s is related to α by

$$d_s = 2 - 2\alpha. \tag{47}$$

Below we will prove a stronger result than (45) and show that there exist positive constants $\underline{c}$ and $\overline{c}$ such that for x small enough

$$\underline{c}x^{-1/3} \leq Q(x) \leq \overline{c}x^{-1/3}. \tag{48}$$

The bounds (48) imply our main result:

Theorem 2 *The spectral dimension of generic infinite tree ensembles equals* 4/3.

In the remainder of this section we prove (48).

4.1 Lower bound on $Q(x)$

In order to establish the relevant lower bound we need two preliminary lemmas.

Lemma 6 *For all finite trees $T \in \Gamma$ and $0 < x \leq 1$ we have*

$$P_T(x) \geq 1 - |T|x. \tag{49}$$

Proof Let $T_1, T_2, \ldots, T_{n-1}$ be the trees attached to the vertex v of T next to the root. Then from (43) we obtain

$$P_T(x) = \frac{1 - x}{n - \sum_{i=1}^{n-1} P_{T_i}(x)}. \tag{50}$$

The first return generating function for the tree consisting of a single link is $1 - x$. The lemma follows by induction on $|T|$. $\square$

Lemma 7 *Let $\tau \in \Gamma$ be a tree with one infinite spine. For all $L \geq 1$ and $0 < x \leq 1$ we have*

$$P_\tau(x) \geq 1 - \frac{1}{L} - Lx - \sum_{T \subset \tau}^{L} (1 - P_T(x)), \tag{51}$$

where $\sum_{T \subset \tau}^{L}$ denotes the sum over all (finite) branches T of τ attached to vertices on the spine at distance $\leq L$ from the root.

Proof We give an inductive argument proving the stronger inequality

$$P_\tau^L(x) \geq 1 - \frac{1}{L} - Lx - \sum_{T \subset \tau}^{L} (1 - P_T(x)), \tag{52}$$

where $P_\tau^L(x)$ denotes the contribution to $P_\tau(x)$ from walks ω that do not visit the vertex s_{L+1} on the spine, i.e. walks constrained to the first L vertices on the spine after the root and the branches attached to them.

The inequality holds for $L = 1$, since the right hand side of (52) is non-positive in this case. For $L \geq 2$ we have from (43)

$$P_\tau^L(x) = \frac{1 - x}{n - P_{\tau_1}^{L-1}(x) - \sum_{k=1}^{n-2} P_{T_k}(x)}, \tag{53}$$

where $n = \sigma_{s_1}$ is the order in τ of the vertex s_1 and $T_1, \ldots, T_{n-2}$ denote the (finite) branches attached to s_1, while τ_1 is the infinite branch attached to s_1, i.e. the subtree with root s_1 and containing s_2 and all its descendants.

Using the induction hypothesis

$$P_{\tau_1}^{L-1}(x) \geq 1 - \frac{1}{L-1} - (L-1)x - \sum_{T \subset \tau_1}^{L-1} (1 - P_T(x)), \tag{54}$$

we get from (53)

$$
\begin{aligned}
P_\tau^L(x) &= \frac{1-x}{1 + (1 - P_{\tau_1}^{L-1}(x)) + \sum_{k=1}^{n-2}(1 - P_{T_k}(x))} \\
&\geq \frac{1-x}{1 + \frac{1}{L-1} + (L-1)x + \sum_{T \subset \tau}^{L}(1 - P_T(x))} \\
&\geq \frac{L-1}{L} \frac{1-x}{1 + (L-1)x + \sum_{T \subset \tau}^{L}(1 - P_T(x))} \\
&\geq (1 - \tfrac{1}{L})(1-x)\left(1 - (L-1)x - \sum_{T \subset \tau}^{L}(1 - P_T(x))\right) \\
&\geq 1 - \frac{1}{L} - Lx - \sum_{T \subset \tau}^{L}(1 - P_T(x)).
\end{aligned}
\tag{55}
$$

Here we have assumed for the last inequality that the final expression is positive. Otherwise, the inequality (54) holds trivially. This proves the lemma. $\qquad\square$

We are now ready to establish the desired lower bound (48) on $Q(x)$. The argument combines Lemmas 3, 6 and 7 with Jensen's inequality. Let s be any vertex on the spine different from r. Given that s has k left branches and ℓ right branches the probability that a given branch has height $> R$ is given by (30). Hence the conditional probability c_R that at least one of the $k + \ell$ branches has height $> R$ fulfills

$$c_R \leq (k + \ell)\left(\frac{2}{f''(1)R} + O(R^{-2})\right). \tag{56}$$

According to (18) the ν-probability q_R that at least one branch at s has height $> R$ then fulfills

$$q_R \leq \left(\frac{2}{f''(1)R} + O(R^{-2})\right) \sum_{k,\ell \geq 0}(k+\ell)\varphi(k,\ell) = \frac{2}{R} + O(R^{-2}). \tag{57}$$

By independence of the distribution of branches attached to different vertices on the spine we conclude that the ν-probability of the event $\mathcal{A}_R$, that all branches attached to the first R vertices $s_1, \ldots, s_R$ on the spine have height $\leq R$, satisfies

$$\nu(\mathcal{A}_R) = (1 - q_R)^R \geq \exp(-2 + O(R^{-1})). \tag{58}$$

Denoting by $\langle \cdot \rangle_R$ the expectation w.r.t. ν conditioned on the event $\mathcal{A}_R$, we get by Lemmas 6 and 7 and Jensen's inequality that

$$Q(x) \geq e^{-2 + O(R^{-1})}\langle(1 - P_\tau(x))^{-1}\rangle_R$$

$$\geq e^{-2+O(R^{-1})}\left\langle \left(\frac{1}{R}+Rx+\sum_{T\subset\tau}^{R}x|T|\right)^{-1}\right\rangle_R$$

$$\geq e^{-2+O(R^{-1})}\left(\frac{1}{R}+Rx+x\left\langle\sum_{T\subset\tau}^{R}|T|\right\rangle_R\right)^{-1}. \tag{59}$$

Let $B_R^i(\tau)$ denote the subgraph of τ which is spanned by all vertices whose distance from the vertex s_i is at most R and which lie in the branches rooted at s_i. Using again that the distributions of branches are identical at all vertices on the spine and given by (18) and (19), we have

$$\left\langle\sum_{T\subset\tau}^{R}|T|\right\rangle_R=\left\langle\sum_{i=1}^{R}|B_R^i(\tau)|\right\rangle_R\leq(1-q_R)^{-1}R\langle|B_R^1|\rangle_v$$

$$=(1-q_R)^{-1}R\sum_{k,\ell\geq0}\varphi(k,\ell)(k+\ell)\langle|B_R|\rangle_\mu=\frac{f''(1)}{1-q_R}R^2. \tag{60}$$

Inserting the last estimate into (59) and observing from (57) that $q_R\to0$ as $R\to\infty$ we deduce

$$Q(x)\geq c'\left(\frac{1}{R}+Rx+f''(1)xR^2\right)^{-1}, \tag{61}$$

where c' is a positive constant, for R large enough. Finally, choosing $R=[x^{-1/3}]$ yields

$$Q(x)\geq\underline{c}x^{-1/3} \tag{62}$$

for a suitable constant $\underline{c}>0$ and x small enough, as claimed.

4.2 Upper bound on $Q(x)$

We begin by establishing a monotonicity lemma which is a slight generalization of the corresponding result in [4].

Lemma 8 *Let τ be a rooted tree, ω_v the shortest path on τ from r to the vertex v and let v_j be the jth vertex from r along ω_v, $j=1,2,\ldots,|\omega_v|$. Denote by $\tau_{j1},\tau_{j2},\ldots,\tau_{jK_j}$ the subtrees of τ attached to the vertex v_j which do not contain any link in ω_v and such that their roots v_j have order 1. Then P_τ is an increasing function of each $P_{\tau_{jk}}$. In particular, if τ' is the tree obtained by removing one of the τ_{jk} from τ, then $P_{\tau'}(x)\geq P_\tau(x)$.*

Proof Let τ_j denote the rooted tree with root v_j obtained from τ by amputating all the branches τ_{ik}, $i=1,2,\ldots,j$, and the links $(r,v_1),(v_1,v_2),\ldots,(v_{j-1},v_j)$. From (43) we have the recursion

$$P_\tau(x)=\frac{1-x}{K_1+2-P_{\tau_1}(x)-\sum_{k=1}^{K_1}P_{\tau_{1k}}(x)}. \tag{63}$$

We see that P_τ is an increasing function of P_{τ_1} and $P_{\tau_{11}},P_{\tau_{12}},\ldots,P_{\tau_{1K_1}}$. The lemma follows by induction. □

The upper bound will be obtained from the above lemma and some elementary estimates. Let $\tau \in \Gamma$. Define $p_\tau(t; v)$ to be the probability that a random walk which starts at the root at time 0 is at the vertex $v \in \tau$ after t steps. That is,

$$p_\tau(t; v) = \sum_{\omega: r \to v} \prod_{s=1}^{t-1} \sigma_{\omega(s)}^{-1}, \tag{64}$$

where the sum is over all walks of length t from r to v. Define the corresponding generating function

$$Q_\tau(x; v) = \sum_{t=0}^{\infty} p_\tau(t; v)(1 - x)^{t/2}, \quad 0 < x \le 1. \tag{65}$$

Note that $Q_\tau(x) = Q_\tau(x; r)$.

Summing (65) over a ball of radius R centred on the root gives

$$\sum_{v \in B_R(\tau)} Q_\tau(x; v) \le \sum_{t=0}^{\infty} (1 - x)^{t/2} \le \frac{2}{x}. \tag{66}$$

It follows that there is a vertex $\bar{v} \in B_R(\tau)$ such that

$$Q_\tau(x; \bar{v}) \le \frac{2}{x |B_R(\tau)|}. \tag{67}$$

If $\bar{v} \ne r$ we can split the random walk representation of $Q_\tau(x)$ into two parts: walks that do not reach the vertex $\bar{v}$ and walks that do. Let us denote the first contribution by $Q_\tau^{(1)}(x)$ and the second one by $Q_\tau^{(2)}(x)$. Let $L \ge 1$ denote the distance of $\bar{v}$ from the root. Then by Lemma 8 we have

$$Q_\tau^{(1)}(x) \le \frac{1}{1 - R_L(x)}, \tag{68}$$

where $R_L(x)$ is the generating function for first return to the root of walks on the integer half line which are restricted not to move beyond the $(L - 1)$st vertex. This function can be calculated explicitly, see [4] Appendix B, with the result

$$R_L(x) = (1 - x) \frac{(1 + \sqrt{x})^{L-1} - (1 - \sqrt{x})^{L-1}}{(1 + \sqrt{x})^L - (1 - \sqrt{x})^L}. \tag{69}$$

It is straightforward to show that

$$\frac{1}{1 - R_L(x)} \le L \tag{70}$$

for $0 < x \le 1$. Hence,

$$Q_\tau^{(1)}(x) \le L \le R. \tag{71}$$

If v, v' are two different vertices in τ we define $G_\tau(x; v, v')$ by (41) with the walks restricted to start at the vertex v, end at v' and not visit v again. Any walk ω that contributes to $Q_\tau^{(2)}(x)$ can be split uniquely into two parts, an arbitrary walk ω_1 from the root to $\bar{v}$ and a walk ω_2 from $\bar{v}$ back to the root which does not revisit $\bar{v}$. Hence,

$$Q_\tau^{(2)}(x) = \sigma_{\bar{v}}^{-1} Q_\tau(x; \bar{v}) G_\tau(x; \bar{v}, r). \tag{72}$$

Let v be any vertex in $\tau \in \Gamma$ different from r. Let ω_v be the shortest path from r to v and $v_0 = r, v_1, \ldots, v_n$ its vertices, $n = |\omega_v|$. Then we have the representation

$$G_\tau(x; v) = \sigma_v(1-x)^{-|\omega_v|/2} \prod_{k=0}^{n-1} P_{\tau_k}(x) \tag{73}$$

which is easily obtained by decomposing a walk ω from r to v into n walks $\omega^k, k = 0, 1, \ldots, n-1$, such that ω^k starts at v_k and ends at v_{k+1} and avoids v_k after leaving it, see [4] Sect. 2.2 for details. In (73) the trees τ_k are defined as in the proof of Lemma 8. Applying (73), with the roles of r and v interchanged, and Lemma 8 we see that

$$G_\tau(x; v, r) \le G_{\tilde\tau}(x; v, r) \tag{74}$$

for any vertex $v \in \tau$ where $\tilde\tau$ is the chain of links forming the shortest path from v to r. It is straightforward to compute $G_{\tilde\tau}(x; v, r)$, see [4], with the result

$$G_{\tilde\tau}(x; v, r) = \frac{2(1-x)^{L/2}}{(1+\sqrt{x})^L + (1-\sqrt{x})^L} \le 1. \tag{75}$$

We conclude from (67), (71) and (72) that

$$Q_\tau(x) \le R + \frac{2}{x|B_R(\tau)|}. \tag{76}$$

Obviously, this inequality also holds if $\bar v = r$. Taking the expectation of the above inequality with respect to the measure ν and using Lemma 5 yields

$$Q(x) \le R + \frac{c}{x R^2} \tag{77}$$

with c a positive constant. This establishes the desired upper bound (48) on $Q(x)$ by choosing $R = [x^{-1/3}]$ and completes the proof of Theorem 1.

Remark 1 The results about the spectral dimension of generic trees generalize to the case when the root r has a fixed order $m \ge 1$. Let $\Gamma^{(m)}$ denote the set of planar trees with a distinguished *root link* (r, r') where r has order m. We define the partition functions $Z_N^{(m)}$, $N \ge m$, by the right hand side of (5) with Γ_N replaced by $\Gamma_N^{(m)} = \{\tau \in \Gamma_N : |\tau| = N\}$. The corresponding generating function $Z^{(m)}(\zeta)$ is then given by $Z^{(m)}(\zeta) = Z(\zeta)^m$ where $Z(\zeta)$ is as in (7). This relation implies an immediate generalization of Lemma 1 and also the existence of a probability measure $\nu^{(m)}$ supported on the subset of $\Gamma^{(m)}$ consisting of trees with one infinite spine originating at r. This measure can be characterized in the same way as ν. In particular, the (finite) branches have the same probability distribution as in the $m = 1$ case and the m branches (including the infinite one) originating at r have equal probabilities of being infinite. Using this observation the arguments of Sect. 4 can be carried through with minor modifications to yield the same upper and lower bounds (48) for the generating function $Q^{(m)}(x)$ for return probabilities to the root r. Similar remarks apply to the case where the root is subject to the same degree distribution as the other vertices of the tree.

5 The Mass Exponent of Generic Random Trees

Let us consider an infinite tree τ with a single spine. We label the vertices on the spine as before $s_0 = r, s_1, s_2, \ldots$. For convenience let us introduce the notation $Q_\tau(x; s_n) = Q_\tau(x; n)$, $n \in \mathbb{N}$. In the same way we write $G(x; s_k, s_n) = G(x; k, n)$ and $G(x; s_n) = G(x; r, s_n) = G(x; n)$. The purpose of this section is to determine the critical behavior of the mass $m(x)$, defined as the exponential decay rate of the *two-point function* $Q(x; n)$, defined by

$$Q(x; n) = \langle Q_\tau(x; n) \rangle_\nu, \tag{78}$$

as a function of n, i.e.,

$$m(x) = -\lim_{n \to \infty} \frac{\log Q(x; n)}{n}. \tag{79}$$

We show below that the limit above exists and is positive for $0 < x < 1$ and tends to zero as $x \to 0$. The critical exponent ν of the mass is defined by

$$m(x) \sim x^\nu \tag{80}$$

as $x \to 0$, provided it exists. The methods of this paper do not suffice to show the existence of ν. We prove that if ν exists then

$$\nu = \alpha = 1/3, \tag{81}$$

where α is defined by (45).

First, let us establish existence of the mass.

Lemma 9 *For a generic ensemble* (Γ, ν) *of infinite trees the mass* $m(x)$ *is well defined by* (79). *It is a non-decreasing function of* x *and fulfills*

$$x^{\frac{1}{2}} e^{-m(x)n} \leq Q(x; n) \leq C \tilde{C}^{-1} x^{-1} e^{-m(x)n} \tag{82}$$

for $n \geq 1$, *where* $C = f''(1) + 2$ *and* $\tilde{C} = \sum_{n,m \geq 0} \frac{\varphi(n,m)}{n+m+2}$.

Proof Note first that, if τ is a rooted tree with a single spine, $Q_\tau(x; n)$ can be written as

$$Q_\tau(x; n) = Q_\tau(x) G_\tau(x; n). \tag{83}$$

Similarly, for $i \geq 1$,

$$G_\tau(x; i) = R_\tau(x; i) G_\tau^0(x; i), \tag{84}$$

where $G_\tau^0(x; i)$ is defined by the same formula (41) as $Q_\tau(x)$ but with the walks ω going from the root to s_i and restricted to visit neither the root nor the endpoint s_i except at their start and end. The function $R_\tau(x; i)$ is defined by the same formula as $Q_\tau(x)$ except that the walks ω start and end at s_i and avoid the root. In the definition of $R_\tau(x; i)$ we also include an extra factor $\sigma_{s_i}^{-1}$ associated with the initial step. It is easy to see that

$$Q_\tau(x) \leq x^{-\frac{1}{2}} \quad \text{and} \quad R_\tau(x; i) \leq \sigma_{s_i} x^{-\frac{1}{2}} \tag{85}$$

using Lemma 8. It follows that

$$G_\tau(x; i) \leq Q_\tau(x; i) \leq x^{-\frac{1}{2}} G_\tau(x; i), \tag{86}$$

and

$$G_\tau^0(x; i) \le G_\tau(x; i) \le \sigma_{s_i} x^{-\frac{1}{2}} G_\tau^0(x; i). \tag{87}$$

Taking averages w.r.t. ν in the above inequalities yields

$$G(x; i) \le Q(x; i) \le x^{-\frac{1}{2}} G(x; i) \tag{88}$$

and

$$G^0(x; i) \le G(x; i) \le C x^{-\frac{1}{2}} G^0(x; i), \tag{89}$$

where $G(x; i)$ and $G^0(x; i)$ denote $\langle G_\tau(x; i) \rangle_\nu$ and $\langle G_\tau^0(x; i) \rangle_\nu$ respectively, and the constant C is given by

$$C = \sum_{n,m=0}^{\infty} (n + m + 2)\varphi(n, m) = f''(1) + 2. \tag{90}$$

By similar arguments, decomposing walks and the tree τ suitably, we have, for $i, j \ge 1$,

$$G_\tau^0(x; i) G_{\tau_i}^0(x; j) \le \sigma_{s_i} G_\tau^0(x; i + j) \le \sigma_{s_i} x^{-\frac{1}{2}} G_\tau^0(x; i) G_{\tau_i}^0(x; j), \tag{91}$$

where τ_i is defined as in the proof of Lemma 8 with ω_v the path along the spine from r to s_i. Averaging w.r.t. ν then yields

$$\tilde{C} G^0(x; i) G^0(x; j) \le G^0(x; i + j) \le x^{-\frac{1}{2}} G^0(x; i) G^0(x; j), \tag{92}$$

since $G_\tau^0(x; i)$ and $G_{\tau_i}^0(x; j)$ are independent random variables.

It follows that $-\log(\tilde{C} G^0(x; i))$ and $\log(x^{-\frac{1}{2}} G^0(x; i))$ are subadditive functions of i. Hence,

$$-\lim_{i \to \infty} \frac{\log G^0(x; i)}{i} = -\sup_{i \ge 1} \frac{\log(\tilde{C} G^0(x; i))}{i} = -\inf_{i \ge 1} \frac{\log(x^{-\frac{1}{2}} G^0(x; i))}{i}. \tag{93}$$

In view of (88) and (89) this proves that the mass exists and fulfills (82). Since $G^0(x; i)$ (as well as $Q(x; i)$) clearly is a decreasing function of x it follows that $m(x)$ is non-decreasing.   □

Theorem 3 *There exist positive constants c_1 and c_2 such that*

$$c_1 \frac{x^{1/3}}{|\log x|^{2/3}} \le m(x) \le c_2 x^{1/3} |\log x| \tag{94}$$

for x sufficiently small.

Proof Let s_n be a vertex on the spine. Then by (68), (72) and (75) we obtain

$$Q_\tau(x) \le \frac{1}{1 - R_n(x)} + \sigma_{s_n}^{-1} Q_\tau(x; n), \tag{95}$$

with $R_n(x)$ as in (69). Taking the expectation value we get

$$Q(x) \le n + Q(x; n). \tag{96}$$

Now choose $n = [|\log x| m(x)^{-1}] + 1$ in (96). In view of (82) we then have

$$Q(x; n) \leq C\tilde{C}^{-1}. \tag{97}$$

Using (62) and (96) we find that as $x \to 0$

$$\underline{c} x^{-1/3} \leq |\log x| m(x)^{-1} + 1 + C\tilde{C}^{-1}, \tag{98}$$

proving the claimed upper bound.

To obtain the lower bound we make use of the representation (73). By (87) it follows that

$$\tilde{C} G^0(x; n) \leq (1 - x)^{-n/2} \prod_{k=0}^{n-1} \langle P_{\tau_k}(x)^n \rangle_\nu^{1/n} = (1 - x)^{-n/2} \langle P_\tau(x)^n \rangle_\nu, \tag{99}$$

where we have used

$$\langle P_{\tau_k}(x)^n \rangle_\nu = \langle P_\tau(x)^n \rangle_\nu \tag{100}$$

which is a consequence of the characterization of ν given in Sect. 2.

By (42) and (76) we have

$$P_\tau(x) \leq 1 - \left(\frac{2}{x |B_R(\tau)|} + R \right)^{-1}. \tag{101}$$

Now, let the event $\mathcal{C}(\lambda, R)$, where $\lambda > 0$ and $R \geq 1$, be defined by

$$\mathcal{C}(\lambda, R) = \{\tau \in \Gamma \,|\, |B_R(\tau)| \geq \lambda R^2\}. \tag{102}$$

It can be shown that

$$\nu(\Gamma \setminus \mathcal{C}(\lambda, R)) \leq e^{-c_0 \lambda^{-1/2}} \tag{103}$$

for some constant $c_0 > 0$ and λ in an interval $(0, \lambda_0)$. This is proven in a special case in [17] Lemma 2.4, but the proof can be generalized in a straightforward manner to arbitrary generic ensembles of infinite trees. For the sake of completeness details are provided in Appendix 2.

Setting $R = [x^{-1/3} |\log x|^{\frac{2}{3}}]$ and $\lambda = c_0^2 / 4 |\log x|^2$ we get for $\tau \in \mathcal{C}(\lambda, R)$ and small x,

$$P_\tau(x) \leq 1 - \frac{c'' x^{1/3}}{|\log x|^{2/3}} \tag{104}$$

where c'' is a positive constant. Furthermore,

$$\nu(\Gamma \setminus \mathcal{C}(\lambda, R)) \leq x^2. \tag{105}$$

Using (99), (104) and (105) we obtain

$$\tilde{C} G^0(x; n) \leq (1 - x)^{-n/2} \left(\left(1 - \frac{c'' x^{1/3}}{|\log x|^{2/3}} \right)^n + x^2 \right). \tag{106}$$

Inserting $n = [2(c'')^{-1} x^{-1/3} |\log x|^{5/3}]$ into this inequality yields

$$\tilde{C} G^0(x; [2(c'')^{-1} x^{-1/3} |\log x|^{5/3}]) \leq 3x^2 \tag{107}$$

for x sufficiently small. Finally, combining (107) with (93), it follows that

$$e^{-nm(x)} \leq 3\tilde{C}^{-1} x^{3/2} \tag{108}$$

implying the lower bound in (94) for x small enough.   □

6 Discussion

It is clear from the arguments in this paper that the existence of a unique spine played a fundamental role. This allowed us to decompose the trees into an infinite line with identically distributed random outgrowths. We believe that our methods will allow one to calculate the spectral dimension of any random tree ensemble with this property. However, nongeneric trees (sometimes called exotic trees) are not expected to have a unique spine in general and it would be interesting to generalize the results of this paper for such trees. For results in this direction, see [15, 21].

Our results are all about ensemble averages. Working slightly harder and using the techniques of [17] we believe that one can obtain estimates of the spectral dimension of generic trees valid with probability one.

In the case of combs [4] we saw that mean field theory holds in the sense that the singularity of the ensemble average of the generating function for first return probabilities was given by that of $Q(x)$, i.e., $Q(x) \sim (1 - P(x))^{-1}$. Whether this holds for trees remains to be seen.

Clearly an outstanding problem is to discover the relevance of loops for the spectral dimension of random graphs. We only expect loops to play a role in determining the spectral dimension if they in a suitable sense bound a large part of the graph. It remains to turn this intuition into a precise statement. Any graph can of course be made a connected tree by removing links and it is not hard to see that cutting links cannot increase the spectral dimension but may decrease it. The role of loops in determining the spectral dimension seems to be the crucial feature that one must understand in order to get a grip on the spectral dimension of planar graphs (see [10] for some relevant results) and higher dimensional random triangulations of interest in quantum gravity.

Appendix 1

In this appendix we provide details of the proof of Theorem 2. It is implicitly assumed below that $Z_N \neq 0$ for all N, but the reader can easily verify that this is not an essential limitation of the arguments.

As explained in [18] it is sufficient to prove, for arbitrary fixed value of $R \geq 1$, that

$$\nu_N(\{\tau \in \Gamma : |B_R(\tau)| > K\}) \to 0 \quad \text{as } K \to \infty \tag{109}$$

uniformly in N, and that the sequence

$$(\nu_N(\{\tau \in \Gamma : B_R(\tau) = \tau_0\}))_{N \in \mathbb{N}} \tag{110}$$

is convergent for each finite tree $\tau_0 \in \Gamma$.

We prove (109) by induction on R, the case $R = 1$ being trivial. Consider first the case $R = 2$. If the volume of $B_2(\tau) = k + 1$ then the order of the vertex s_1 next to the root is $k + 1$. Letting N_i, $i = 1, 2, \ldots, k$, denote the volumes of the various subtrees attached to s_1 we have the following estimate,

$$
\nu_N(\{\tau \in \Gamma : |B_2(\tau)| = k + 1\}) = Z_N^{-1} w_{k+1} \sum_{N_1 + \cdots + N_k = N-1} \prod_{i=1}^{k} Z_{N_i}
$$

$$
\leq k \zeta_0 w_{k+1} \sum_{\substack{N_1 + \cdots + N_k = N-1 \\ N_1 \geq (N-1)/k}} \frac{Z_{N_1} \zeta_0^{N_1}}{Z_N \zeta_0^{N}} \prod_{i=2}^{k} (Z_{N_i} \zeta_0^{N_i})
$$

$$
\leq C k^{5/2} w_{k+1} Z_0^{k-1} \tag{111}
$$

for $k \geq 1$, where we have used (13), and $C > 0$ is a constant independent of k and N. Since $Z_0 < \rho$ the last expression tends to 0 as $k \to \infty$, proving (109) for $R = 2$. Note that this inequality holds also for $k = 0$ if $k^{5/2}$ is replaced by $(k + 1)^{5/2}$.

Assume (109) holds for some $R \geq 2$. Since the set of different balls $B_R(\tau)$ of volume at most K is finite for each fixed $K > 0$, it suffices to show that

$$
\nu_N(\{\tau : |B_{R+1}(\tau)| > K, B_R(\tau) = \tau_0\}) \to 0 \quad \text{as } K \to \infty \tag{112}
$$

uniformly in N for every finite tree τ_0 of height R. Define

$$
A_L(N, \tau_0) = \nu_N(\{\tau : |B_{R+1}(\tau)| = L, B_R(\tau) = \tau_0\}). \tag{113}
$$

Let M be the number of vertices in τ_0 at distance R from the root and let their orders be $k_1, k_2, \ldots, k_M$. Now we decompose any tree τ with $B_R(\tau) = \tau_0$ into τ_0 and subtrees whose root is at distance $R - 1$ from the root in τ. Note that the root link of these subtrees overlaps with τ_0, see Fig. 2. Then we have the following formula

$$
A_L(N, \tau_0) = Z_N^{-1} \sum_{\substack{k_1 + \cdots + k_M = L - |\tau_0| \\ N_1 + \cdots + N_M = N + M - |\tau_0|}} \left(\prod_{i=1}^{M} Z_{N_i}^{(k_i)} \right) \prod_{i \in B_{R-1}(\tau_0) \backslash r} w_{\sigma_i}, \tag{114}
$$

Fig. 2 The tree τ_0 in the case $R = 4$. The circles denote finite (possibly empty) trees that are attached to the vertices at distance R
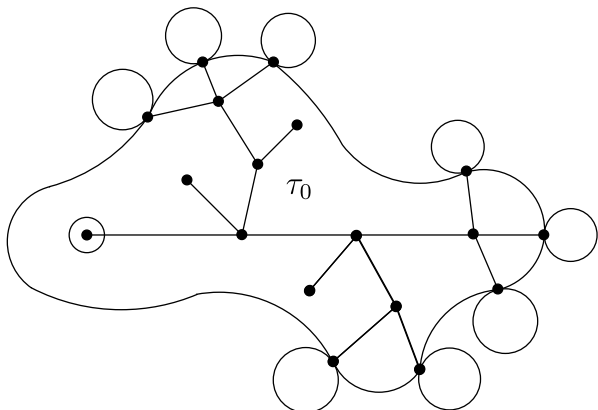
where $Z_N^{(k)}$ is the contribution to Z_N from trees whose vertex next to the root has order $k+1$, that is

$$Z_N^{(k)} = Z_N \nu_N(\{\tau : |B_2(\tau)| = k+1\}). \tag{115}$$

Denoting the last product in (114) by $W(\tau_0)$ and using (111) and (115) we get

$$A_L(N, \tau_0) \leq W(\tau_0) \sum_{\substack{k_1+\cdots+k_M \\ =L-|\tau_0|}} \left(\prod_{i=1}^{M} C(k_i+1)^{5/2} w_{k_i+1} Z_0^{k_i-1} \right)$$

$$\times \sum_{\substack{N_1+\cdots+N_M \\ =N+M-|\tau_0|}} \left(Z_N^{-1} \prod_{j=1}^{M} Z_{N_j} \right). \tag{116}$$

The last sum can be estimated as in (111) with the result

$$A_L(N, \tau_0) \leq W(\tau_0) C^{M+1} M^{5/2} Z_0^{M-1}$$

$$\times \sum_{k_1+\cdots+k_M=L-|\tau_0|} \prod_{i=1}^{M} (k_i+1)^{5/2} w_{k_i+1} Z_0^{k_i-1}, \tag{117}$$

provided N is large enough. Summing over L from $K+1$ to ∞ we obtain

$$\nu_N(\{|B_{R+1}(\tau)| > K, B_R(\tau) = \tau_0\})$$

$$\leq W(\tau_0) C^{M+1} M^{5/2} Z_0^{-1} \sum_{k_1+\cdots+k_M>K-|\tau_0|} \prod_{i=1}^{M} (k_i+1)^{5/2} w_{k_i+1} Z_0^{k_i}$$

$$\leq W(\tau_0) C^{M+1} M^{7/2} Z_0^{-1} \left(\sum_{k=1}^{\infty} k^{5/2} w_k Z_0^{k-1} \right)^{M-1}$$

$$\times \left(\sum_{k>(K-|\tau_0|)/M} k^{5/2} w_k Z_0^{k-1} \right). \tag{118}$$

We have generic trees so the sum $\sum_{k=1}^{\infty} k^{5/2} w_k Z_0^{k-1}$ converges and (112) follows from (118) since the last term in (118) tends to 0 as $K \to \infty$.

It remains to verify (110). Summing over L in (114) we get

$$\nu_N(\{B_R(\tau) = \tau_0\}) = W(\tau_0) Z_N^{-1} \sum_{\substack{k_1,\ldots,k_M \geq 0 \\ N_1+\cdots+N_M=N+M-|\tau_0|}} \prod_{i=1}^{M} Z_{N_i}^{(k_i)}$$

$$= W(\tau_0) Z_N^{-1} \sum_{N_1+\cdots+N_M=N+M-|\tau_0|} \prod_{i=1}^{M} Z_{N_i}. \tag{119}$$

Now choose a constant A. By arguments identical to those of [18] p. 4804 one shows that the contribution to the sum in (119) from terms where $N_i \geq (N+M-|\tau_0|)/M$ and $N_j \geq A$ for some pair of indices $i \neq j$ can be estimated from above by *constant* $\cdot A^{-1/2}$ uniformly

in N. Note that the condition on N_i is always satisfied for at least one i. The remaining contribution is, for sufficiently large N,

$$W(\tau_0) \sum_{i=1}^{M} \sum_{\substack{N_1+\cdots+N_M=N+M-|\tau_0| \\ N_j \leq A, j \neq i}} Z_N^{-1} \prod_{i=1}^{M} Z_{N_i}. \tag{120}$$

Letting $N \to \infty$ with fixed A, the last expression converges to

$$M W(\tau_0) \left(\sum_{N=1}^{A} Z_N \zeta_0^N \right)^{M-1} \zeta_0^{|\tau_0|-M}, \tag{121}$$

by Lemma 1. Letting $A \to \infty$ we conclude that

$$\nu_N(\{B_R(\tau)=\tau_0\}) \xrightarrow[N\to\infty]{} M W(\tau_0) Z_0^{M-1} \zeta_0^{|\tau_0|-M}, \tag{122}$$

which proves (110).

Note that the above estimates show that for any constant A, all the N_j's except one are bounded by A, with a probability which tends to 1 as $A \to \infty$, while one of them gets very large when N gets very large. A slight generalization of the arguments leading to (122) (see [18]) shows that ν is concentrated on the set of infinite trees with a single spine and that ν can be characterized as explained in Sect. 2.

Appendix 2

In this appendix we establish the inequality (103), that is

$$\nu(\{\tau : |B_R(\tau)| < \lambda R^2\}) \leq e^{-c_0 \lambda^{-1/2}}, \quad \text{for } R > 0 \text{ and } 0 < \lambda < \lambda_0, \tag{123}$$

where c_0 and λ_0 are positive constants.

Given $\tau \in \Gamma$ with one infinite spine, let $B_R^i(\tau)$ denote the intersection of the ball of radius R, centered at the ith spine vertex s_i, with the finite branches attached to the spine at s_i. Since

$$B_{[R/2]}^1(\tau) \cup \cdots \cup B_{[R/2]}^{[R/2]}(\tau) \subseteq B_R(\tau), \tag{124}$$

it is sufficient to prove

$$\nu(\{\tau : |B_R^1| + \cdots + |B_R^R| < \lambda R^2\}) \leq e^{-c_0 \lambda^{-1/2}}. \tag{125}$$

Let us fix $i \in \mathbb{N}$ and set $|B_R^i| = Y_R$. Then

$$Y_R = X_1 + \cdots + X_R, \tag{126}$$

where $X_n(\tau)$ is the number of vertices in $B_n^i(\tau)$ at distance n from s_i.

From the properties of ν given in Sect. 2 and the discussion of Sect. 3 it follows that the generating functions

$$h_n(z) = \sum_{a=0}^{\infty} \nu(\{\tau : X_n(\tau) = a\})z^a,$$

$$k_n(z) = \sum_{b=0}^{\infty} \mu(\{T : D_n(T) = b\})z^b, \tag{127}$$

with D_n as in Sect. 3, are related by

$$h_n(z) = f'(k_n(z)). \tag{128}$$

Since $\nu(\{X_n > 0\}) = 1 - h_n(0)$ and $\mu(\{D_n > 0\}) = 1 - k_n(0)$ it follows from (30) that

$$\nu(\{X_n > 0\}) = \frac{2}{n} + O(n^{-2}). \tag{129}$$

Similarly, the generating functions

$$g_R(z) = \sum_{a=0}^{\infty} \nu(\{\tau : Y_R = a\})z^a,$$

$$f_R(z) = \sum_{b=1}^{\infty} \mu(\{T : |B_R(T)| = b\})z^b \tag{130}$$

are related by

$$g_R(z) = f'(f_R(z)). \tag{131}$$

Here f_R is determined by the recursion relation

$$f_{R+1}(z) = zf(f_R(z)), \qquad f_1(z) = z. \tag{132}$$

Differentiating and evaluating at $z = 1$, one easily deduces from these two relations that

$$\langle |B_R| \rangle_\mu = R, \tag{133}$$

$$\langle Y_R \rangle_\nu = f''(1)R, \tag{134}$$

$$\langle |B_R|^2 \rangle_\mu = \frac{1}{3}f''(1)R^3 + O(R^2), \tag{135}$$

$$\langle Y_R^2 \rangle_\nu = \frac{1}{3}f''(1)^2 R^3 + O(R^2). \tag{136}$$

Using (129), (134) and (136) we now prove, following [17], that

$$\nu(\{Y_R \geq c'R^2\}) \geq \frac{c''}{R}, \tag{137}$$

for suitable positive constants c', c''. For this purpose let $\mathcal{A}_n = \{\tau : X_{[n/2]} > 0\}$. Then, by (129),

$$\nu(\mathcal{A}_n) = \frac{4}{n} + O(n^{-2}). \tag{138}$$

Since $Y_n = Y_{[n/2]}$ on the complement of $\mathcal{A}_n$, we have by (134),

$$f''(1)(n - [n/2]) = \int_{\mathcal{A}_n} (Y_n - Y_{[n/2]})d\nu \leq \int_{\mathcal{A}_n} Y_n d\nu, \tag{139}$$

and therefore

$$\langle Y_n | \mathcal{A}_n \rangle_\nu \geq \frac{f''(1)n}{2\nu(\mathcal{A}_n)} = \frac{f''(1)n^2}{8} + O(n) \geq c_1 n^2, \tag{140}$$

where $\langle \cdot | \mathcal{A}_n \rangle_\nu$ denotes the expectation w.r.t. ν conditioned on $\mathcal{A}_n$. We also have

$$\langle Y_n^2 | \mathcal{A}_n \rangle_\nu \leq \frac{\langle Y_n^2 \rangle_\nu}{\nu(\mathcal{A}_n)} = \frac{1}{12} f''(1)^2 n^4 + O(n^3) \leq c_2 n^4, \tag{141}$$

by (136) and (138). Combining the last two inequalities with the reversed Chebyshev inequality for the conditional expectation $\langle \cdot | \mathcal{A}_n \rangle_\nu$,

$$\nu(\mathcal{A}_n)^{-1}\nu(\{Y_n \geq \tfrac{1}{2}\langle Y_n | \mathcal{A}_n \rangle_\nu\} \cap \mathcal{A}_n) \geq \frac{\langle Y_n | \mathcal{A}_n \rangle_\nu^2}{4\langle Y_n^2 | \mathcal{A}_n \rangle_\nu}, \tag{142}$$

see e.g. [22] p. 152, we obtain

$$\nu\left(\left\{Y_n \geq \tfrac{1}{2}c_1 n^2\right\}\right) \geq \nu\left(\left\{Y_n \geq \tfrac{1}{2}c_1 n^2\right\} \cap \mathcal{A}_n\right) \geq \frac{c_1^2}{4c_2}\nu(\mathcal{A}_n) \geq \frac{c''}{n}, \tag{143}$$

which proves (137).

Finally, (125) can be obtained from (137) as follows. Given $\lambda \in (0, c')$ let $n = [(\lambda/c')^{1/2}R]$. Then $n < R$ and

$$\{|B_R^1| + \cdots + |B_R^R| \leq \lambda R^2\} \subseteq \{|B_n^1| + \cdots + |B_n^R| \leq \lambda R^2\}$$
$$\subseteq \{|B_n^1| + \cdots + |B_n^R| \leq c' n^2\}$$
$$\subseteq \{|B_n^i| \leq c' n^2, i = 1, \ldots, R\}. \tag{144}$$

Hence, using (137) and noting that $|B_n^i|, i = 1, \ldots, R$, are independent random variables, we obtain

$$\nu(\{|B_R^1| + \cdots + |B_R^R| \leq \lambda R^2\}) \leq \left(1 - \frac{c''}{n}\right)^R \leq \left(1 - \frac{c'^{1/2}c''\lambda^{-1/2}}{R}\right)^R. \tag{145}$$

This proves (125) with $c_0 = c'^{1/2}c''$ and $\lambda_0 = c'$.

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